Functions

Derivation (part 1)

Introduction

The derivative is the main tool of Differential Calculus. Specifically, a derivative is a function that tells us about *rates of change*, or *slopes* of tangent lines.
Its definition involves limits.
Suppose that we have the function $f(x) = 2x^5 + 7x^3 + 5$.
Through a process called differentiation we can find another function that is related to f. This second function called derivative of f. for this example the derivative is:

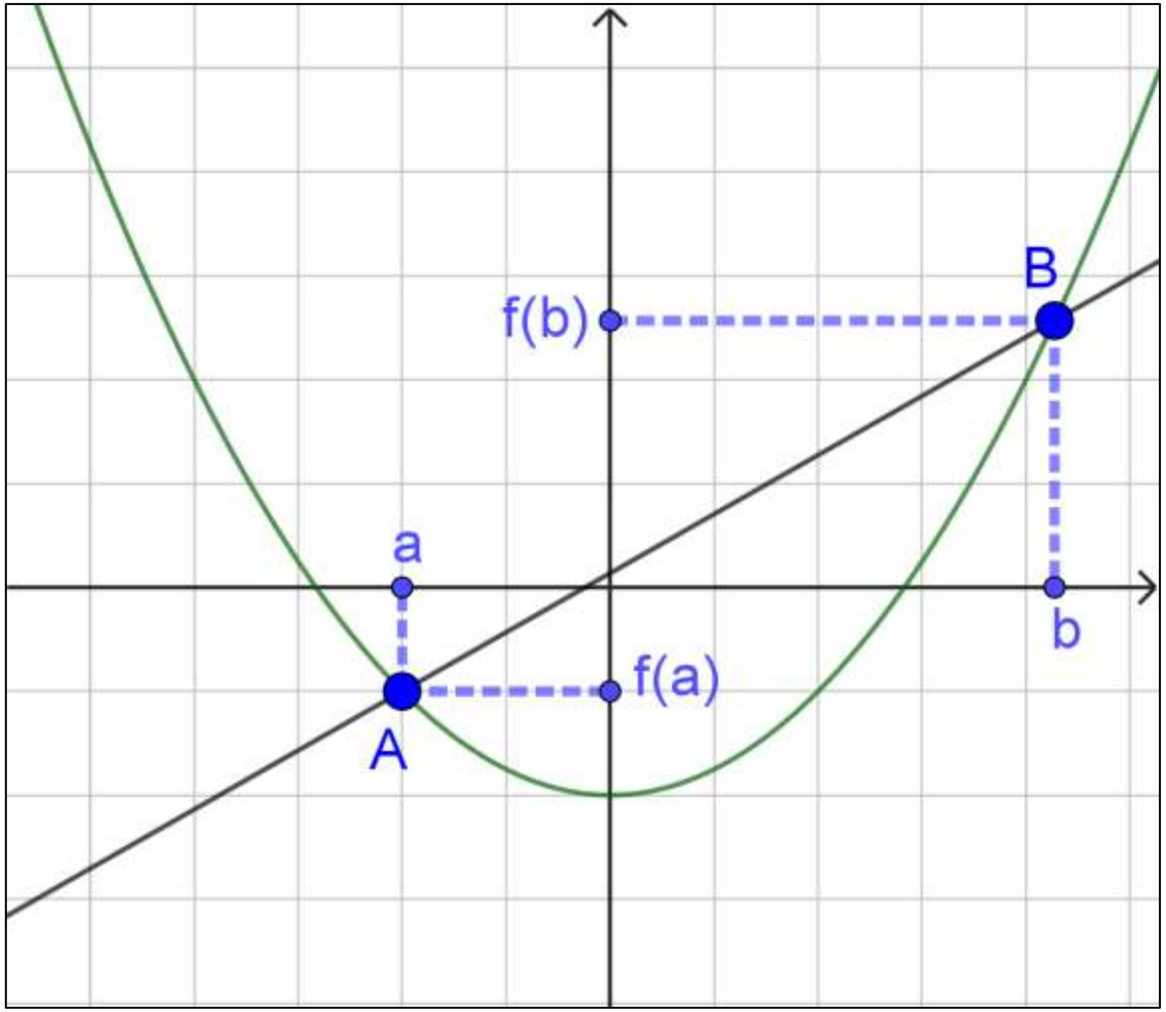
$$f'(x) = 10x^4 + 21x^2$$

Definition 1

Average rate

Let f be a function defined over an interval I. a and b are two elements of I.

The average rate of change is the real number $\frac{f(b)-f(a)}{b-a}$ which is the slope of the line (AB).



Definition 2

Let f be a function defined over I. the derivative of f at a (an element of I), denoted by f'(a) is defined by:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Lagrange notation

Remark:
There is another notation of the derivative, called Leibnitz notation or ratio notation: $\frac{df}{dx}$

Example

Using the definition of the derivative of a function f at a point a, calculate the derivative in each of the following cases.

❶ $f(x) = x^2 - 2x + 1 \quad ; \quad a = 2$

$$\lim_{x\to 2}\frac{f(x)-f(2)}{x-2} = \lim_{x\to 2}\frac{x^2-2x+1-(4-4+1)}{x-2} = \lim_{x\to 2}\frac{x^2-2x}{x-2} = \lim_{x\to 2}\frac{x(x-2)}{x-2} = \lim_{x\to 2} x$$

$= 2$ so $f'(2) = 2$

❷ $f(x) = \sqrt{x+1} \quad ; \quad a = 1$

$$\lim_{x\to 1}\frac{f(x)-f(1)}{x-1} = \lim_{x\to 1}\frac{\sqrt{x+1}-\sqrt{2}}{x-1}$$ by rationalize

$$= \lim_{x\to 1}\frac{x+1-2}{(x-1)(\sqrt{x+1}+\sqrt{2})} = \lim_{x\to 1}\frac{1}{\sqrt{x+1}+\sqrt{2}} = \frac{1}{2\sqrt{2}}$$ so $f'(1) = \frac{1}{2\sqrt{2}}$

Definition 3

Differentiability

f is differentiable at $x = a$ if and only if:

- ❖ f is continuous at $x = a$
- ❖ $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exist: $f'_-(a) = f'_+(a)$

Where $f'_-(a) = \lim_{x \to a^-} \frac{f(x) - f(a)}{x - a}$; $f'_+(a) = \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a}$

Example 1

$f(x) = x^2$

Study the differentiability of f at $x = 1$.

f is continuous at $x = 1$

$$f'(1) = \lim_{x\to 1} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x-1)(x+1)}{x-1} = \lim_{x\to 1} x+1 = 2$$

So f is differentiable at $x = 1$

Example 2

$$f(x) = \begin{cases} 2x & if\ x < 1 \\ x + 1 & if\ x > 1 \\ 2 & if\ x = 1 \end{cases}$$

Study the differentiability of f at $x = 1$.

$f(1) = 2$

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 2x = 2$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x + 1 = 1 + 1 = 2$ so f is continuous at x=1.

$\lim_{x \to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x \to 1^-} \frac{2x-2}{x-1} = \lim_{x \to 1^-} \frac{2(x-1)}{x-1} = 2$

$\lim_{x \to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x \to 1^+} \frac{x+1}{x-1} = \frac{2}{0^+} = +\infty$ so f is not differentiable at x=1

Graphical interpretation

Consider the points A(a,f(a)) and M$(x, f(x))$.

(T) Is the tangent to (C) at A

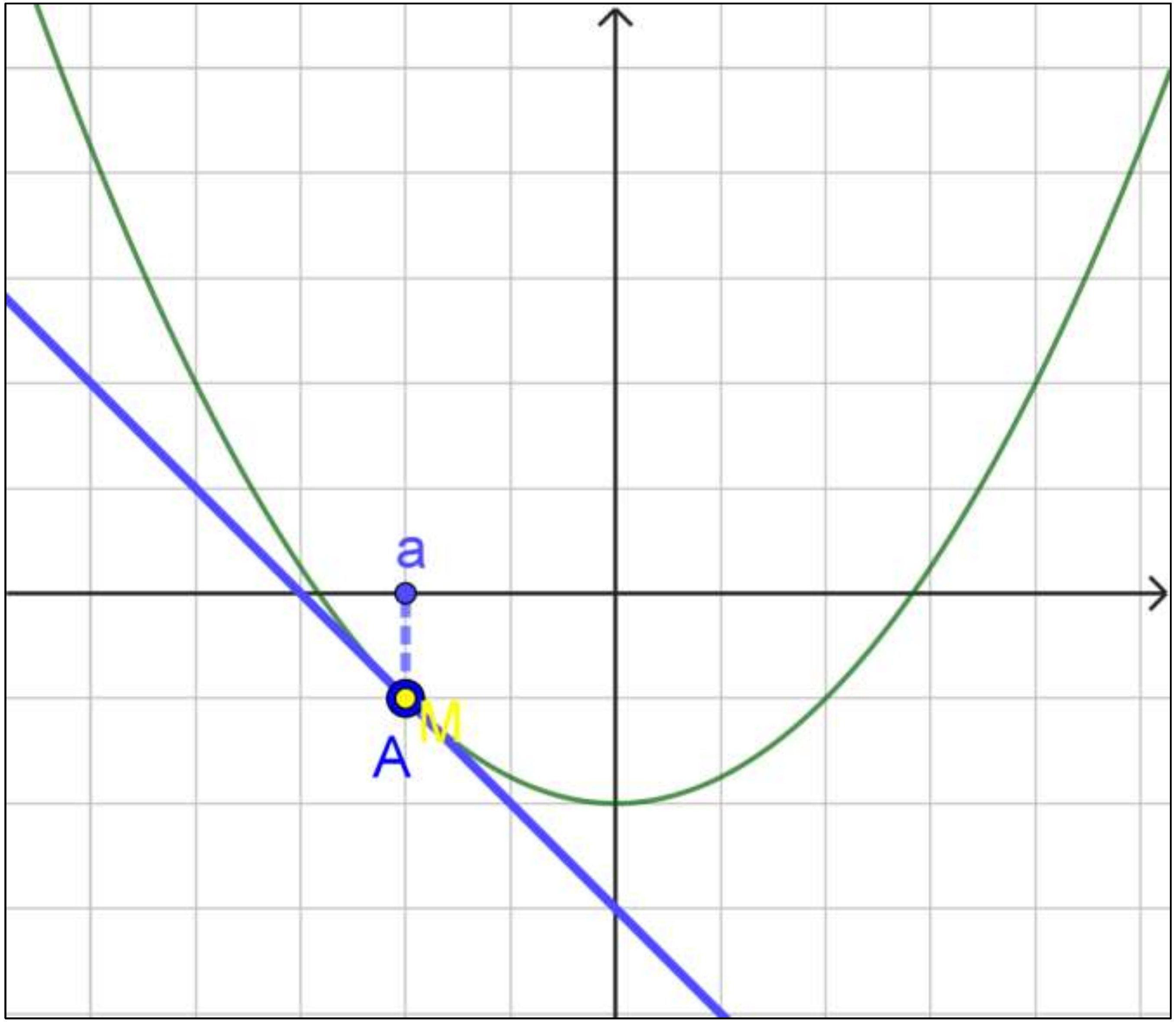
The slope of the line (AM) is:

$$m = \frac{y_M - y_A}{x_M - x_A} = \frac{f(x) - f(a)}{x - a}$$

As $x \to$ a

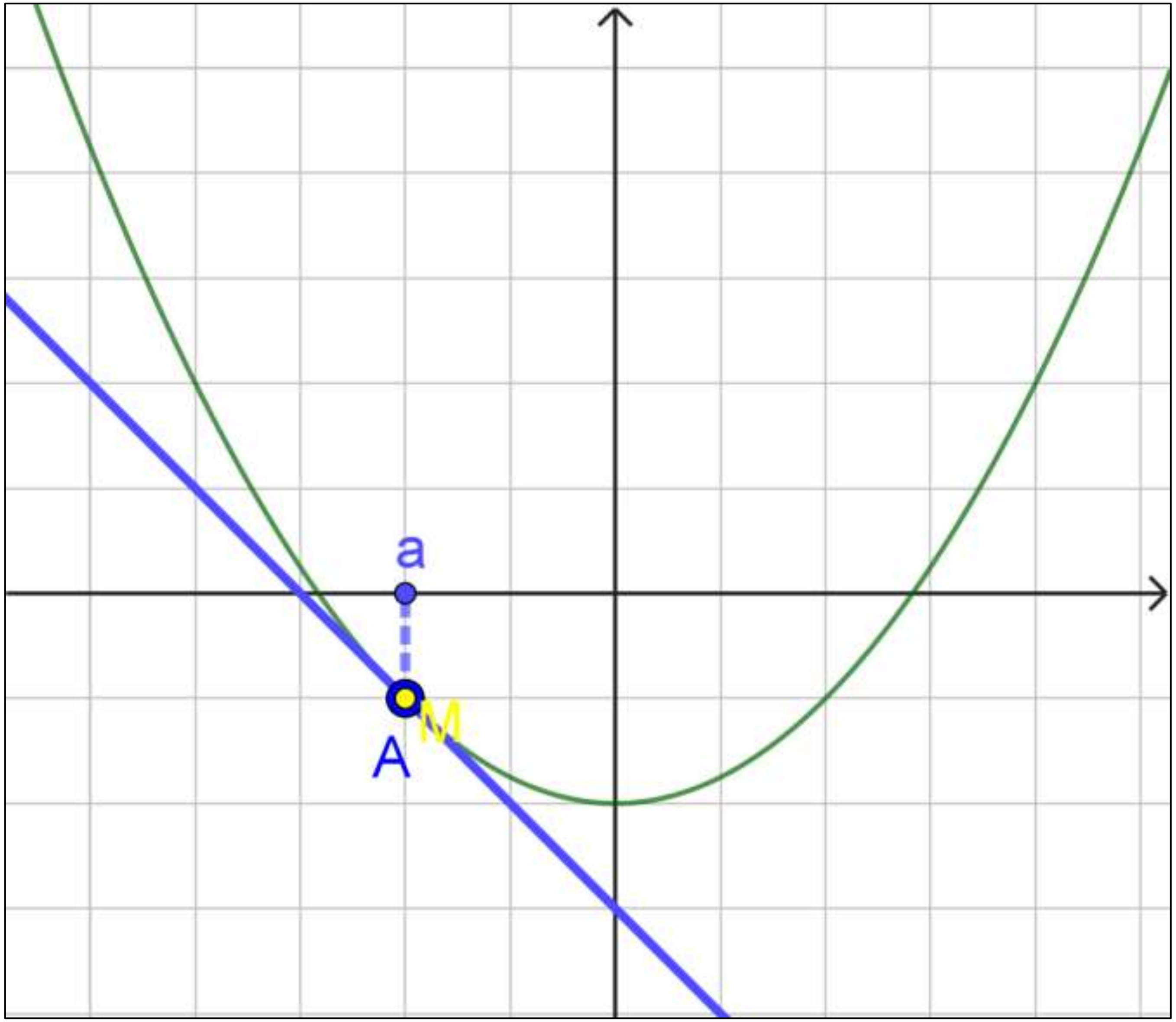
The line (AM) tends to the tangent at A

So $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ is the slope of (T)



Graphical interpretation

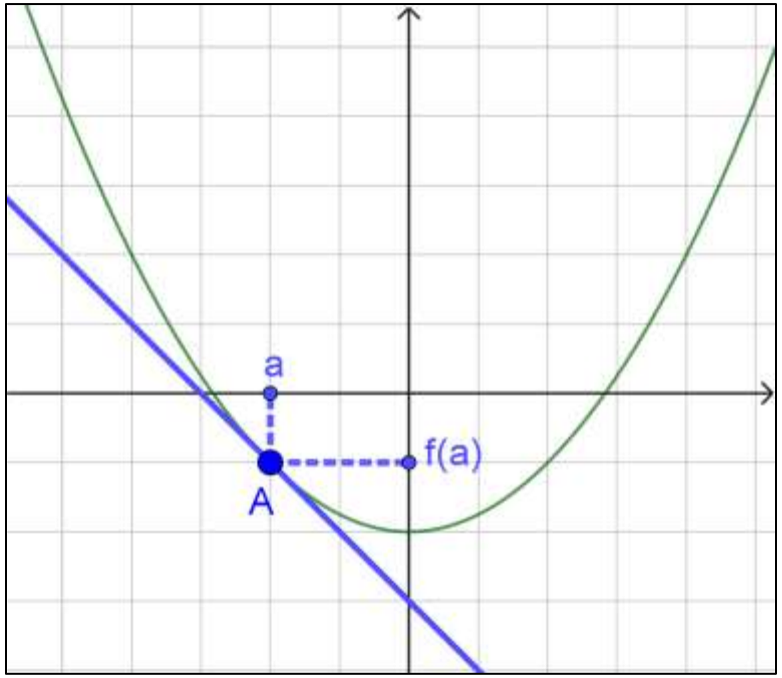
As a result, $f'(a) =$ slope of the tangent to the curve at point of abscissa a



Graphical interpretation

Equation of tangent

$$y = f'(a)(x - a) + f(a)$$



Example:

$f(x) = \frac{x^2}{4}$; $x = -2$

$f(-2) = \frac{(-2)^2}{4} = 1$

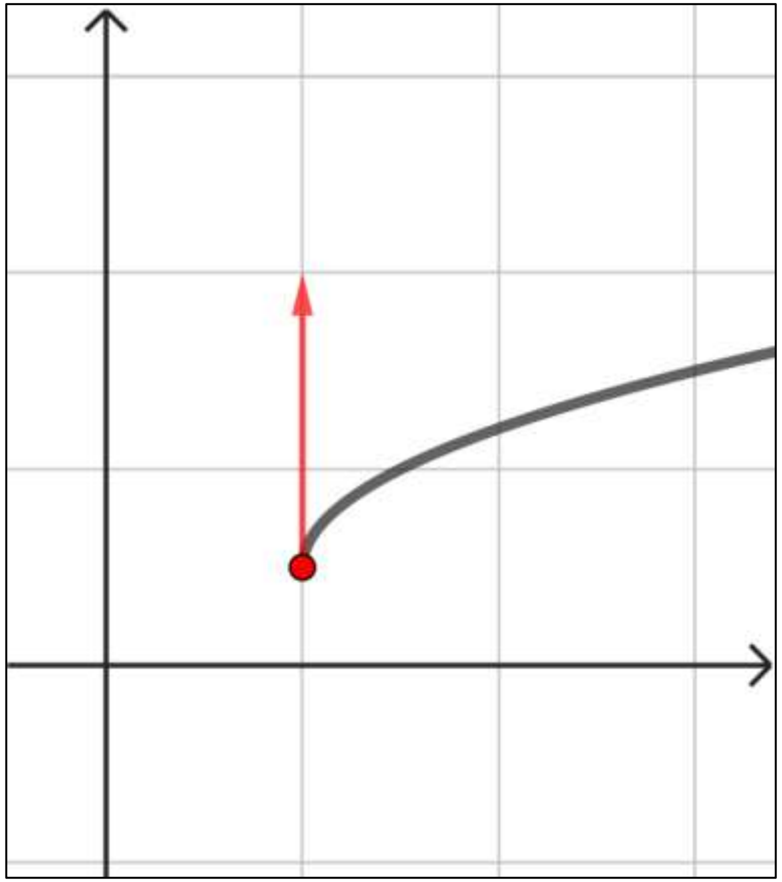
$f'(-2) = \lim_{x \to -2} \frac{\frac{x^2}{4} - 1}{x+2} = \lim_{x \to -2} \frac{x^2 - 4}{4(x+2)} = \lim_{x \to -2} \frac{(x-2)(x+2)}{4(x+2)} = \lim_{x \to -2} \frac{x-2}{4} = -\frac{4}{4}$

$= -1$

So the equation of the tangent is $y = -1(x + 2) + 1 = -x - 1$

Differentiability (graphically)

Case 1

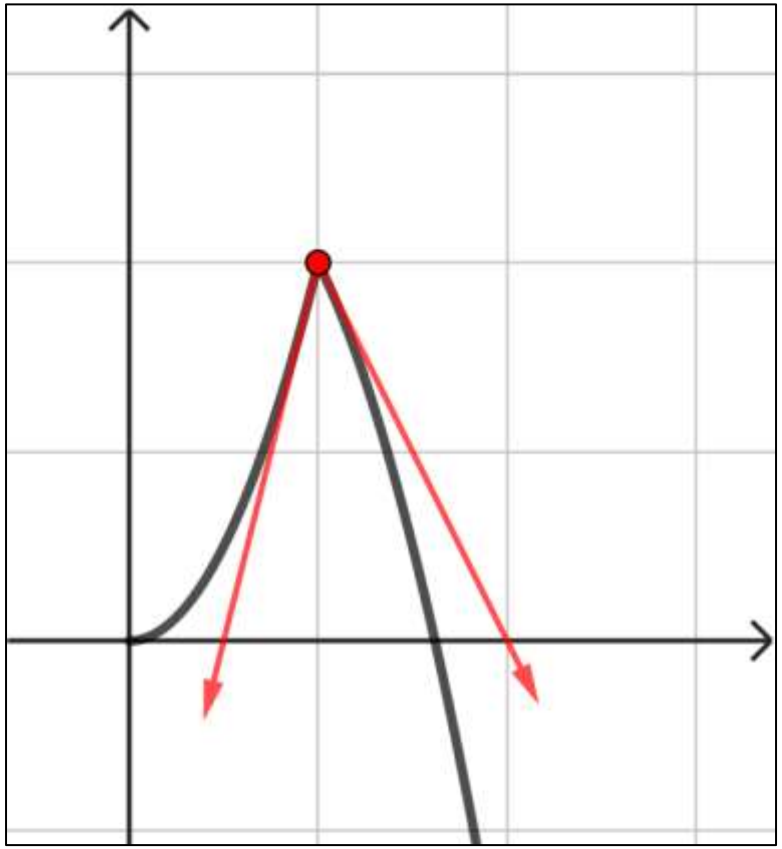


In this case, the tangent is vertical
Slope is not defined

Not differentiable

Differentiability (graphically)

Case 2

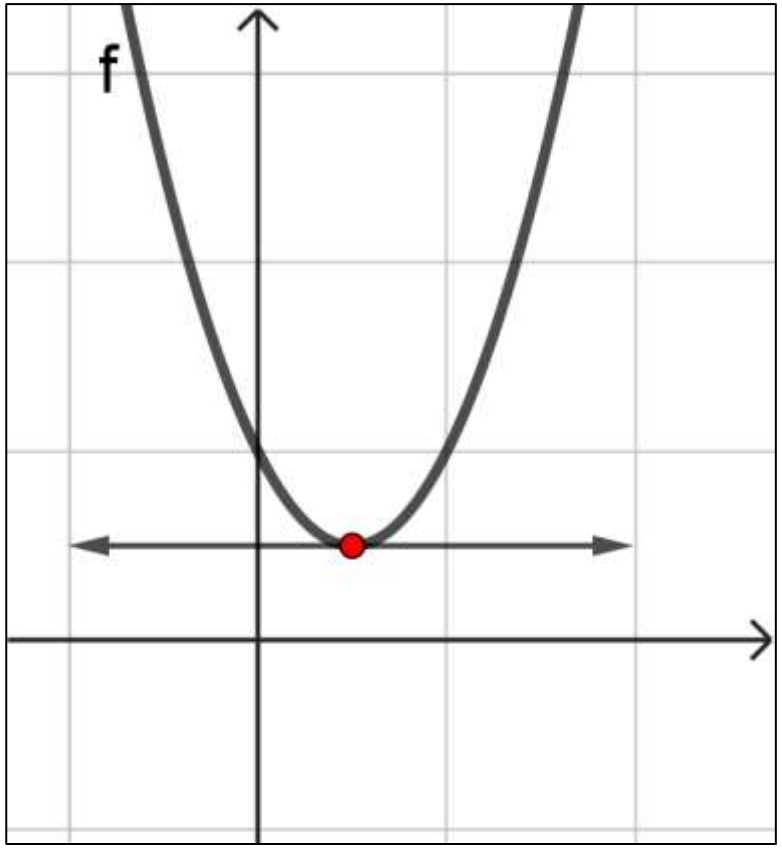


In this case, at the angular point (sharp point) there is two semi tangent.
In this case $f'_{-}(a) \neq f'_{+}(a)$

Not differentiable

Differentiability (graphically)

Case 3

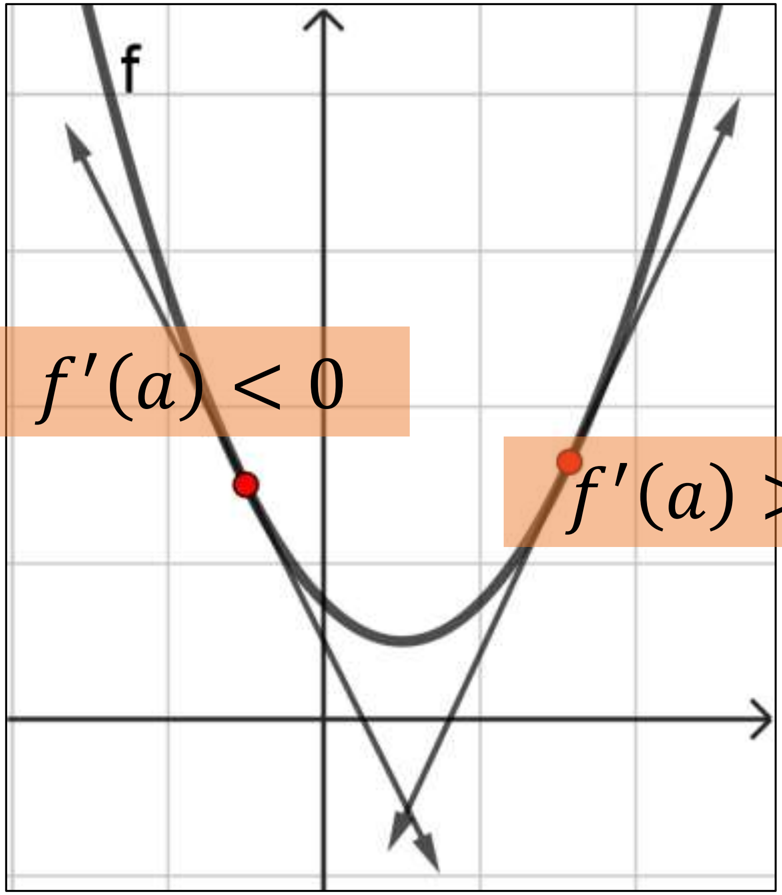


In this case, the tangent is horizontal of equation y=f(a) and of slope 0

Differentiable

Differentiability (graphically)

Case 4



$f'(a) < 0$

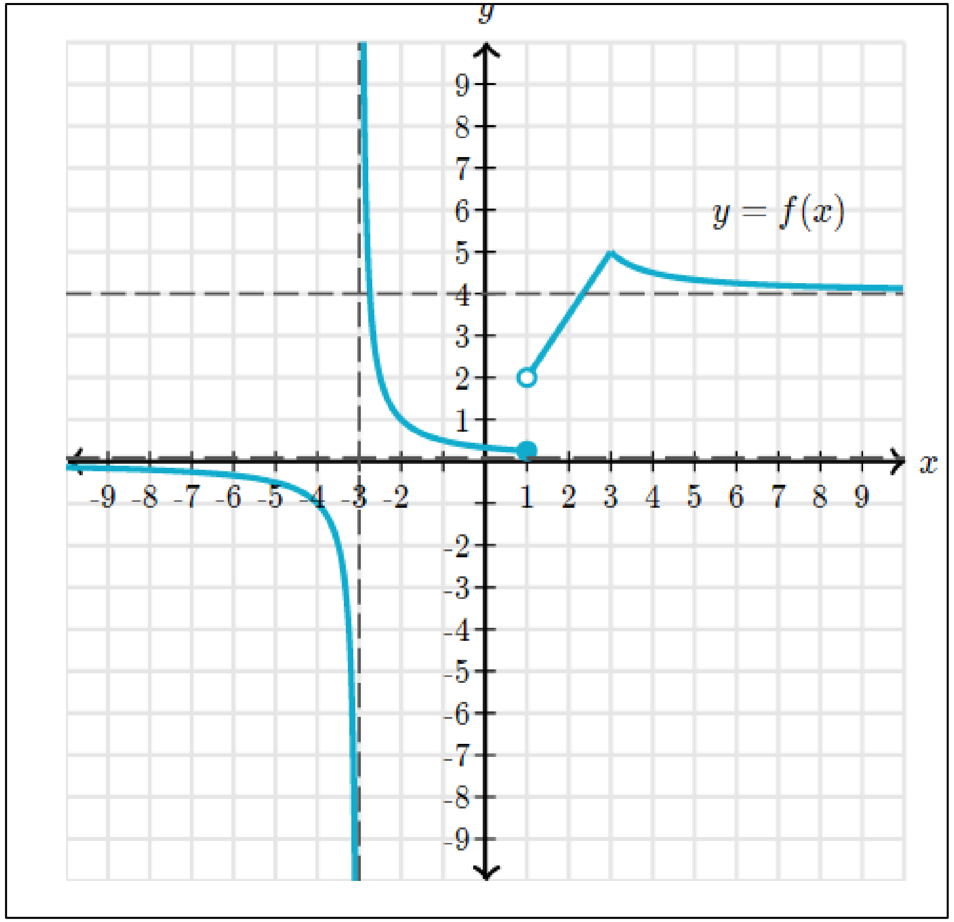
$f'(a) > 0$

In the two cases, the tangent is in the form of y=ax+b and is unique in each case.

Differentiable

Differentiability (graphically)

Case 5



In this case, the function is not continuous at x=1 and x=-3

Not differentiable

Application

Give a point for which:

a) f is differentiable.

(0,0) since f is continuous at 0 and there is no sharp point

b) f is continuous and not differentiable.

(1,4) since it is a sharp point

c) f is differentiable and not continuous.

Impossible case since continuity is one of the condition of differentiability